Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

# *Note on the General Solution of Bessel's Equation.*

By A. Chessin.

The general solution of Bessel's equation

$$\frac{d^2z}{dx^2} + \frac{1}{x}\,\frac{dz}{dx} + \left(1 - \frac{n^2}{x^2}\right)z = 0,$$

when $n$ is neither an integer nor zero, as we know, is

$$z = AJ_n(x) + BJ_{-n}(x),$$

where

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{\kappa=0}^{\kappa=\infty} \frac{(-1)^\kappa \left(\frac{x}{2}\right)^{2\kappa}}{\kappa!\,(n+\kappa)!}, \tag{1}$$

and $J_{-n}(x)$ is exactly the same function of $-n$ as $J_n(x)$ is of $n$. If $n$ is an integer, which we can suppose to be positive, the general solution of Bessel's equation is

$$z = AJ_n(x) + BK_n(x), \tag{2}$$

where*

$$K_n(x) = J_n(x)\log x - \frac{1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{\kappa=0}^{\kappa=n-1} \frac{(n-\kappa-1)!}{\kappa!}\left(\frac{x}{2}\right)^{2\kappa} - \frac{1}{2}S_n(x), \tag{3}$$

$$S_n(x) = S_{n1}(x) + S_{n2}(x), \tag{4}$$

$$S_{n1}(x) = \left(\frac{x}{2}\right)^n \sum_{\kappa=0}^{\kappa=\infty} \frac{(-1)^\kappa \left(\frac{x}{2}\right)^{2\kappa}}{\kappa!\,(n+\kappa)!}\left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{\kappa}\right], \tag{5}$$

$$S_{n2}(x) = \left(\frac{x}{2}\right)^n \sum_{\kappa=0}^{\kappa=\infty} \frac{(-1)^\kappa \left(\frac{x}{2}\right)^{2\kappa}}{\kappa!\,(n+\kappa)!}\left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+\kappa}\right]. \tag{6}$$

---

* *Vide* M. Bocher, Ann. Math., Vol. VI, No. 4 ; W. Byerly, An elementary treatise on Fourier's series and spherical, cylindrical, etc., harmonics, p. 220.

We will give to the sum $S_n(x)$ the form of a definite integral. To that purpose remark that

$$1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{\kappa}\quad=\int_0^1\frac{1-y^{\kappa}}{1-y}\,dy,$$

$$1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n+\kappa}=\int_0^1\frac{1-y^{n+\kappa}}{1-y}\,dy.$$

Therefore,

$$S_{n1}(x)=\left(\frac{x}{2}\right)^n\sum_{\kappa=0}^{\kappa=\infty}\left\{\frac{(-1)^{\kappa}\left(\frac{x}{2}\right)^{2\kappa}}{\kappa!\,(n+\kappa)!}\int_0^1\frac{1-y^{\kappa}}{1-y}\,dy\right\}$$

$$=\left(\frac{x}{2}\right)^n\int_0^1\frac{dy}{1-y}\left\{\sum_{\kappa=0}^{\kappa=\infty}\frac{(-1)^{\kappa}\left(\frac{x}{2}\right)^{2\kappa}}{\kappa!\,(n+\kappa)!}-\sum_{\kappa=0}^{\kappa=\infty}\frac{(-1)^{\kappa}\left(\frac{x\sqrt{y}}{2}\right)^{2\kappa}}{\kappa!\,(n+\kappa)!}\right\}$$

$$=\int_0^1\frac{J_n(x)-y^{-n/2}J_n(x\sqrt{y})}{1-y}\,dy,$$

and in like manner

$$S_{n2}(x)=\int_0^1\frac{J_n(x)-y^{n/2}J_n(x\sqrt{y})}{1-y}\,dy.$$

Hence

$$S_n(x)=\int_0^1\frac{2J_n(x)-(y^{n/2}+y^{-n/2})\,J_n(x\sqrt{y})}{1-y}\,dy. \tag{7}$$

When $n=0$,

$$K_0(x)=J_0(x)\log x-\int_0^1\frac{J_0(x)-J_0(x\sqrt{y})}{1-y}\,dy. \tag{8}$$